REAL ANALYSIS: HOMEWORK 2

LAURA LYMAN

(1) Let μ be a Borel measure on $[0,1]$ with $\mu([0,1]) = 1$. Then there exists a compact set $K \subseteq [0,1]$ such that $\mu(K) = 1$ but $\mu(H) < 1$ for any proper compact subset H of K. We call K the *support* of μ. Then every compact subset of [0,1] is the support of some Borel measure.

Proof. Let:

$$U = \bigcup_{\mu(I_k)=0} I_k \qquad \text{for } I_k = (p,q) \text{ with } p,q \in \mathbb{Q} \cap [0,1].$$

Let $K = U^c \cap [0,1]$ so $K \subseteq [0,1]$ and $\mu(K) = 1 - \mu(U) = 1 - 0 = 1$. Note that K is closed (since $K^c = U \cup [0,1]^c$ which is open) and bounded, so K is compact.

Consider $H \subsetneq K$ such that H is compact. Let I_k be a rational interval contained in $K \setminus H$. Suppose $\mu(I_k) = 0$. Then $I_k \subseteq U \subseteq (U \cup [0,1]^c) = K^c$, which is a contradiction since $I_k \subseteq K$. Thus, $\mu(I_k) > 0$. However, we then have that

$$\mu(H) = 1 - \mu([0,1] \setminus H)$$

and since $I_k \subseteq ([0,1] \setminus H)$, $0 < \mu(I_k) \leq \mu([0,1] \setminus H)$. Hence, $\mu(H) < 1$, as desired.

Now suppose K is an arbitrary compact subset of $[0,1]$ (i.e. not the carefully constructed one above). We will show that K is the support of some Borel measure. Since K is compact, K is separable; hence, K contains a countable dense subset $\{k_n\}_{n\in\mathbb{N}}$. Consider the Dirac measure $\delta_x(S) = \begin{cases} 1 \text{ if } x \in S \\ 0 \text{ else.} \end{cases}$ Then for $A \subseteq [0,1]$ we define a measure, as kindly suggested during office hours, by

$$\mu(A) = \sum_{n=1}^{\infty} \frac{1}{2^n} \delta_{k_n}(A).$$

First we show that this is indeed a measure. The largest $\mu(A)$ can be is if $\delta_{k_n}(A) = 1$ for all k_n, in which case $\mu(A) = \sum_{n=1}^{\infty} \frac{1}{2^n} = 1 < \infty$ (thus the measure is well-defined). Clearly the measure is nonnegative and $\mu(\emptyset) = 0$. Now suppose $A = \cup_{\ell=1}^{\infty} A_\ell$. Then

$$\mu(A) = \sum_{n=1}^{\infty} \frac{1}{2^n} \delta_{k_n}\big(\bigcup_{\ell=1}^{\infty} A_\ell\big) \leq \sum_{n=1}^{\infty} \frac{1}{2^n}\big(\sum_{\ell=1}^{\infty} \delta_{k_n}(A_\ell)\big) = \sum_{\ell=1}^{\infty}\sum_{n=1}^{\infty} \frac{1}{2^n}\delta_{k_n}(A_\ell) = \sum_{\ell=1}^{\infty} \mu(A_\ell)$$

where we used the fact that $\delta_{k_n}(A)$ is a measure in the first inequality. Hence, μ is a Borel measure.

Furthermore, $\mu(K) = \sum_{n=1}^{\infty} \frac{1}{2^n} = 1$. Now suppose $H \subsetneq K$ is compact. Then some $k_n \notin H$ (see *Lemma 1*), meaning $\delta_{k_n}(H) = 0$ for that k_n. Hence, $\mu(H) < \sum_{n=1}^{\infty} \frac{1}{2^n}(1) = 1$, as desired. Thus, K is the support of this measure. □

Lemma 1. Let K, H be compact subsets of $[0,1]$ with $H \subsetneq K$. Let $\{k_n\}$ be a countable dense subset of K. Then $k_n \notin H$ for some $n \in \mathbb{N}$.

Proof. Suppose for contradiction that $k_n \in H$ for all $n \in \mathbb{N}$. Let $x \in K \setminus H$. Since $x \in K$ and $\{k_n\}$ is a dense subset of K, either some k_n equals x or x is a limit point of $\{k_n\}$. Since each $k_n \in H$ and $x \notin H$, we have that x must be a limit point of $\{k_n\}$. However, since H is compact in $\mathbb{R}$, H is closed and therefore contains all of its limit points; thus $x \in H$. This is a contradiction. $\square$

(2) Construct a function f such that each set $\{x : f(x) = \alpha\}$ is measurable for any $\alpha \in \mathbb{R}$ but $\{x : f(x) > 0\}$ is *not* measurable.

Proof. First, we construct the *Vitali set* $V \subseteq [0,1]$, which is not measurable. Partition [0,1] into a disjoint union of equivalence classes under the relation $x \sim y \Leftrightarrow x - y \in \mathbb{Q}$. (Note that this is possible since $\mathbb{Q}$ is a normal subgroup of $\mathbb{R}$). Using the Axiom of Choice, we can select a representative element from each equivalence class and put these representatives into a set V.

Suppose V is measurable. Enumerate $\mathbb{Q}$ as $\{q_i\}$ and let

$$U_{q_i} = q_i/\mathbb{Z} + V.$$

Then $\mu(U_{q_i}) = \mu(V)$ by translation invariance. Furthermore, the U_{q_i} are disjoint because no two elements in V have a rational difference. Finally, $\cup_{i=1}^{\infty} U_{q_i} = [0,1]$. However, observe that by countable subadditivity,

$$\mu(\bigcup_{i=1}^{\infty} U_{q_i}) = \sum_{i=1}^{\infty} \mu(U_{q_i}) = \sum_{i=1}^{\infty} \mu(V),$$

meaning $\sum_{i=1}^{\infty} \mu(V) = 1$. If $\mu(V) > 0$, then $1 = \infty$ and if $\mu(V) = 0$, then $1 = 0$. This is a contradiction. Thus, we have shown the existence of an unmeasurable subset V of $[0,1]$.

Now define $f : [0,1] \rightarrow \mathbb{R}$ as

$$f(x) = \begin{cases} x \text{ if } x \in V \\ -x \text{ if } x \notin V \end{cases}$$

meaning that $\{x : f(x) = \alpha\} \subseteq \{\alpha, -\alpha\}$. Since $\{x : f(x) = \alpha\}$ is contained in a discrete set, $\{x : f(x) = \alpha\}$ is clearly measurable.

Now consider $f^{-1}(0,\infty)$ and note that $V^c \cap f^{-1}(0,\infty) = \emptyset$. To see this, observe that if $x \in f^{-1}(0,\infty)$ then $x \in (0,1]$ and if also $x \in V^c$ then $f(x) = -x$. However, $-x < 0$ so $f(x) < 0$; hence, $x \notin f^{-1}(0,\infty)$ which is a contradiction.

Therefore, we have that $f^{-1}(0,\infty) \subseteq V \sqcup V^c$ but $f^{-1}(0,\infty) \cap V^c = \emptyset$, meaning $f^{-1}(0,\infty) \subseteq V$. Since $V \subseteq f^{-1}(0,\infty)$ as well, we have by mutual containment that $\{x : f(x) > 0\} = V$, which is not measurable. $\square$

(3) Construct a monotone function that is discontinuous on a dense set on $[0,1]$.

Proof. Enumerate $\mathbb{Q}$ as $\{q_n\}_{n=1}^{\infty}$. Let $f : [0,1] \longrightarrow \mathbb{R}$ be a function such that:

$$f(x) = \sum_{q_n \leq x} \frac{1}{2^n}.$$

Note that this does *not* assume the enumeration of the rationals preserves the ordering of the reals (that is, this does *not* say that $q_m < q_n \Leftrightarrow n < m$). Since $\sum_{n=1}^{\infty} \frac{1}{2^n}$ is convergent, we have no convergence issues so f is well-defined. We now show that f is monotone. Suppose $x < y$ for $x, y \in [0,1]$. Consider the interval (x,y) and note that some $q_i \in (x,y)$. Thus, $q_i \nleq x$ and $q_i \leq y$, so $\sum_{q_n \leq x} \frac{1}{2^n} < \sum_{q_n \leq y} \frac{1}{2^n}$. Hence, $f(x) < f(y)$ so f is strictly monotone increasing.

We now show that f is discontinuous on $\mathbb{Q} \cap [0,1]$, meaning f is discontinuous on a dense set in $[0,1]$. Let $x_k = q_n - (1/2)^k$ so $\{x_k\} \to q_n$ from below; let the sequence indexing begin at k large enough such that $x_k \in [0,1]$ (so $f(x_k)$ is defined). Since $x_k < q_n$ for all k and f is strictly monotone increasing, $f(x_k) < f(q_n)$. Therefore, $\lim_{k\to\infty} f(x_k) < \lim_{k\to\infty} f(q_n) = f(q_n)$, meaning f is *not* continuous at q_n. (Really, letting $\{x_k\}$ be any sequence in [0,1] approaching q_n from below without ever reaching q_n would have been sufficient). Since $q_n \in \mathbb{Q}$ was arbitrary, f is discontinuous on $\mathbb{Q} \cap [0,1]$. □

(4) Let ϕ be a non-negative continuous function on $\mathbb{R}^n$ such that $\int \phi = 1$. Given $t > 0$, define $\phi_t(x) = t^{-n}\phi(x/t)$. Then if $g \in C^\infty(\mathbb{R}^n)$ with compact support,

$$\phi_t(g) = \int_{R^n} \phi_t(x)g(x)dx \longrightarrow g(0).$$

As a result, ϕ_t is called an *approximation of identity.* How much can you weaken the regularity assumptions on ϕ and g?

Proof. Letting $y = xt$, we have by a change of variables that

$$\int_{\mathbb{R}^n} \phi_t(x)dx = \int_{\mathbb{R}^n} \phi_t(y/t)t^n dy = \int_{\mathbb{R}^n} \phi(y)t^{-n}t^n dy = \int_{\mathbb{R}^n} \phi(y)dy.$$

Since $\int_{\mathbb{R}^n} \phi(y)dy = 1$, we have that $\int_{\mathbb{R}^n} \phi_t(x)dx = 1$ as well. Thus,

$$\begin{aligned}
\left|\int_{\mathbb{R}^n} \phi_t(x)g(x)dx - g(0)\right| &= \left|\int_{\mathbb{R}^n} \phi_t(x)g(x)dx - \int_{\mathbb{R}^n} \phi_t(x)g(0)dx\right| \\
&= \left|\int_{\mathbb{R}^n} \phi_t(x)(g(x)-g(0))dx\right| \\
&\leq \int_{\mathbb{R}^n} |\phi_t(x)||g(x)-g(0)|dx \\
&= \int_{\mathbb{R}^n} \phi_t(x)|g(x)-g(0)|dx \qquad \text{[since } \phi \text{ nonnegative} \Rightarrow \phi_t \text{ nonnegative]} \\
&= \int_{\mathbb{R}^n} \phi(x)|g(xt)-g(0)|dx \qquad \text{[changing variables } x \to xt\text{]}
\end{aligned}$$

Let $M = \sup_{x\in\mathbb{R}^n} |g(x)|$, which exists since $g \in \mathbb{C}^\infty(\mathbb{R}^n)$ with compact support. Then

$$\phi(x)|g(tx) - g(0)| \leq \phi(x)(|g(tx)| + |g(0)|) \leq \phi(x)(2M)$$

meaning $\phi(x)|g(tx) - g(0)|$ is dominated by an integrable function. Furthermore, since g is continuous at zero, $\lim_{t\to 0}(g(xt) - g(0)) = 0$. Thus, by the Dominated Convergence theorem,

$$\lim_{t\to 0} \int_{\mathbb{R}^n} \phi(x)|g(tx) - g(0)|dx = \int_{\mathbb{R}^n} \lim_{t\to 0} \phi(x)|g(tx) - g(0)|dx = \int_{\mathbb{R}^n} \phi(x)(0)dx = 0.$$

Hence, we have that as $t \to 0$, $\int_{\mathbb{R}^n} \phi_t(x)g(x)dx \to 0$ as desired.

We can weaken several of the conditions. We require that ϕ is nonnegative and $\int \phi = 1$ (ϕ does not need to be continuous). In addition, we just need g to be bounded and continuous at zero (we do not need $g \in \mathbb{C}^\infty(\mathbb{R}^n)$ with compact support). □

(5) Let E_k be a sequence of measurable sets such that

$$\sum_{k=1}^{\infty} \mu(E_k) < \infty.$$

Then almost all x lie in at most finitely many of the sets E_k.

Proof. Let V be the set of all x such that x lies in infinitely many of the E_k. That is,

$$V = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} E_k.$$

Let $A_n = \bigcup_{k=n}^{\infty} E_k$ and note that $A_1 \supseteq A_2 \supseteq \cdots$ is a sequence of nested sets. By Proposition 1.31 in the notes,

$$\begin{aligned} \mu(V) &= \lim_{n\to\infty} \mu(A_n) = \lim_{n\to\infty} \mu(\bigcup_{k=n}^{\infty} E_k) \\ &\leq \lim_{n\to\infty} \sum_{k=n}^{\infty} \mu(E_k) = \lim_{n\to\infty} (\sum_{k=1}^{\infty} \mu(E_k) - \sum_{k=1}^{n-1} \mu(E_k)) \\ &= (\sum_{k=1}^{\infty} \mu(E_k) - \sum_{k=1}^{\infty} \mu(E_k)) = 0. \end{aligned}$$

Thus, V has measure zero. Note that V^c is the set of all x contained in at most finitely many of the E_k. Thus, almost all x are contained in at most finitely many of the E_k. □